$$U(\xi) = \frac{105}{169}\,\mathrm{ch}^{-4}(\xi/\sqrt{52})$$

Other values of the constants yield solutions of the Kawahara equation in the form of periodic waves.

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NON-AXISYMMETRIC BUCKLING OF SHALLOW SPHERICAL SHELLS*

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The buckling of elastic shallow orthotropic spherical shells subjected to a transverse load is investigated on the basis of geometrically non-linear equilibrium equations in a non-axisymmetric formulation. By using the method of finite differences and a continuation procedure in the prameters in combination with a Newton operator method an algorithm is constructed to determine the state of shell stress and strain in the pre- and post-critical stages.

The upper critical loads (CL) of spherical shells are determined for different external pressure distribution laws taking perturbing factors such as initial harmonic and azimuthal imperfection directions in the shape of the shell middle surface and analogous load deviations

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from a uniformly distributed load into account. Under the imperfections mentioned, good agreement is obtained with the results for the upper CL found by the theory of buckling and the initial post-critical behaviour /1-4/. Special attention is paid to an investigation of the non-axisymmetrical buckling of an isotropic spherical shell closed at the apex. It is shown that the presence of small initial imperfections is the reason for a substantial reduction in the upper critical load and, moreover, its values can be determined by the formula for unimodal buckling not at the least bifurcation point but at the one following if the initial damage component proportional to the harmonic natural mode of this second bifurcation point is predominant.

1. ***Formulation of the problem.*** The equations of geometrically non-linear elastic shells of transversally-orthotropic truncated spherical shells with initial damage can be written in dimensionless variables in the form /5/

$$A\,(S_1,\ S_2,\ S_3,\ w) - [w - z,\ F] = T\,(p,\ x,\ \theta) \tag{1.1}$$

$$A\,(S_4, S_5, S_6, F) - \left[z - \frac{1}{2}\,w, w\right] = 0$$

$$A\,(S_1, S_2, S_3, w) = S_1\left(w^{\mathrm{IV}} + \frac{2}{x}\,w^{\mathrm{III}}\right) + S_2\left[\frac{1}{x^4}\,w^{\cdot\cdot\cdot\cdot} + \frac{2}{x^4}\,w^{\cdot\cdot} - \right.$$

$$\left.\frac{1}{x}\left(\frac{1}{x}\,w'\right)'\right] + S_3\left(\frac{1}{x^2}\,w''^{\cdot\cdot} - \frac{1}{x^3}\,w'^{\cdot\cdot} + \frac{1}{x^4}\,w^{\cdot\cdot}\right)$$

$$[w,\ F] = l_1wl_2F + l_1Fl_2w - 2l_3wl_3F,\ l_1w = w''$$

$$l_2w = \frac{1}{x}\,w' + \frac{1}{x^2}\,w^{\cdot\cdot},\quad l_3w = \frac{1}{x^2}\,w^{\cdot} - \frac{1}{x}\,w'^{\cdot},\quad (\ )' = \frac{\partial}{\partial x}(\ ),\quad (\ )^{\cdot} = \frac{\partial}{\partial\theta}(\ )$$

$$z\,(x,\ \theta) = z_*\,(x) + \xi\zeta\,(x,\ \theta),\ z_*\,(x) = (\Lambda^2 - x^2)/2,\ 0 \leqslant \theta < 2\pi$$

We will consider the system (1.1) together with the boundary conditions

$$[F = F' = \Gamma_3w = \Gamma_4w]_{x=\Lambda_0} = 0 \tag{1.2}$$

$$[w = w' = \Gamma_1F = \Gamma_2F]_{x=\Lambda} = 0 \tag{1.3}$$

$$\Gamma_1F = F'' + S_7l_2F,\quad \Gamma_2F = xF''' - S_7\left(\frac{1}{x}\,F' - \frac{1}{x}\,F'^{\cdot\cdot} + \frac{2}{x^2}\,F^{\cdot\cdot}\right) +$$

$$S_8\left(\frac{1}{x}\,F'^{\cdot\cdot} - \frac{1}{x^2}\,F^{\cdot\cdot}\right) - S_9l_2F,\quad \Gamma_3w = w'' + S_{10}l_2w$$

$$\Gamma_4w = xw''' + w'' - S_{11}l_2w + S_{12}\left(\frac{1}{x}\,w'^{\cdot\cdot} - \frac{1}{x^2}\,w^{\cdot\cdot}\right)$$

The dimensionless quantities in (1.1)-(1.3) are related to the dimensional ones by the formulas

$$w = \frac{W}{2H\varepsilon^2},\quad F = \frac{\Phi}{4E_1H^2h\varepsilon^4},\quad z = \frac{1}{2H\varepsilon^2}\left\{H\left[1 - \left(\frac{\rho}{a}\right)^2\right] + \xi\zeta_0\,(\rho, \theta)\right\}$$

$$x = \frac{\rho}{a\varepsilon},\quad T = \frac{Xa^4}{8E_1H^3h\varepsilon^2},\quad \varepsilon^2 = \frac{h}{4H\sqrt{3\,(1 - \nu^2)}},\quad \Lambda = \varepsilon^{-1},\quad \Lambda_0 = \frac{a_0}{a\varepsilon}$$

Here W is the deflection, Φ is the Airy stress function, X is the transverse load intensity, and ρ, θ are polar coordinates. The function $\xi\zeta_0\,(\rho,\ \theta)$ describes the initial deflection of the spherical shell with middle surface $H\,[1 - (\rho/a)^2]$ where ξ is a scalar parameter, a is the radius of the reference contour, a_0 is the radius of a circular hole with centre at the point $\rho = 0$, and H is the rise of a corresponding spherical segment. The boundary conditions (1.2) correspond to a free edge for $\rho = a_0$, and (1.3) to a clamped edge for $\rho = a$.

By using the relationships in /6/ the constants S_i can be written in the form

$$S_1 = \frac{1 - \nu^2}{1 - \nu_1\nu_2},\quad S_2 = \frac{1 - \nu^2}{k_1 - \nu_1^2},\quad S_3 = 2\,(1 - \nu^2)\left(\frac{\nu_1}{k_1 - \nu_1^2} + \frac{2}{k_1k_2}\right), \tag{1.4}$$

$$S_4 = k_1,\quad S_5 = 1,\quad S_6 = k_1k_2 - 2\nu_1,\quad S_7 = -S_{10} = -\frac{\nu_1}{k_1},\quad S_8 = k_2,$$

$$S_9 = S_{11} = \frac{1}{k_1},\quad S_{12} = \frac{\nu_1}{k_1} + \frac{4\,(1 - \nu_1^2k_1^{-1})}{k_1k_2},\quad k_1 = \frac{E_1}{E_2},\quad k_2 = \frac{E_2}{G},$$

$$E_1\nu_2 = E_2\nu_1$$

Here E_1, E_2, ν_1, ν_2, G are, respectively, the Young's moduli, Poisson's ratios, and the shear modulus.

A relatively small number of papers /7-17/ consider the direct numerical computations of the non-linear behaviour of spherical shells taking non-axisymmetric strains into account. The solution of the initial boundary-value problem for an isotropic spherical shell is reduced in these papers to the solution of a system of non-linear algebraic equations. For this all the dependent variables in /9/ are sought in the form of cosine series in the azimuthal direction, and the system of ordinary differential equations obtained for the coefficients is then discretized by using the method of finite differences. The Galerkin method is used in /10/ with a two-parameter basis. The system of non-linear algebraic equations is derived in /11-17/ by the method of finite differences in a two-dimensional mesh in a polar system of coordinates.

The present paper is among the last group. Unlike preceding investigations a new algorithm is proposed here for calculating the state of stress and strain of a shell in the post-critical stage, and the upper CL is determined on the basis of a finite-difference analogue of the buckling criterion.

2. ***Application of the method of finite differences.*** We will solve the boundary-value problem (1.1)-(1.3) by finite differences. Assuming the state of stress and strain to be symmetrical about a plane drawn perpendicular to the plane of the shell base through the ray $\theta = 0$ /12/, we separate the domain $D = \{\Lambda_0 \leqslant x \leqslant \Lambda,\ 0 \leqslant \theta \leqslant \pi\}$ into N equal parts along the radial coordinate x and into M equal parts along the angular coordinate θ. Consequently, we obtain the finite difference mesh $(x_\alpha,\ \theta_\gamma)$, where $x_\alpha = \Lambda_0 + \alpha h,\ \theta_\gamma = \gamma\Delta\theta,\ h = (\Lambda - \Lambda_0)/N,\ \Delta\theta = \pi/M$. We introduce the $2(N+1)(M+1)$ dimensional column vector

$$Y = (y_0,\ y_1,\ \ldots,\ y_M),\ y_\gamma = (w_{0\gamma},\ F_{0\gamma},\ w_{1\gamma},\ F_{1\gamma},\ \ldots,\ w_{N\gamma},\ F_{N\gamma})$$
$$w_{\alpha\gamma} = w(x_\alpha,\ \theta_\gamma),\ \gamma = 0,\ 1,\ \ldots,\ M;\ \alpha = 0,\ 1,\ \ldots,\ N$$

that is formed by the manifold of values of the pair of functions $w,\ F$ on the rays $\theta = \theta_\gamma$ at the mesh nodes. We replace the partial derivatives of the functions in system (1.1) by known central finite-difference formulas by using a 13-point pattern. We have introduce nodes outside the contour with the coordinates

$$(x_\alpha,\ \theta_\gamma),\ \alpha = -2,\ -1,\ N+1,\ N+2,\ \gamma = -1,\ 0,\ \ldots,\ M+1$$
$$(x_\alpha,\ \theta_\gamma),\ \alpha = 0,\ 1,\ \ldots,\ N,\ \gamma = -2,\ -1,\ M+1,\ M+2$$
$$\theta_{-i} = -i\Delta\theta,\ x_{-i} = \Lambda_0 - ih,\ \theta_{M+i} = \pi + i\Delta\theta$$
$$x_{N+i} = \Lambda + ih,\ i = 1,\ 2$$

to write central differences on the arcs $x = \Lambda,\ \ x = \Lambda - h,\ x = \Lambda_0,\ \ x = \Lambda_0 + h\ \ (0 < \theta < \pi)$ and on the rays $\theta = 0,\ \theta = \Delta\theta,\ \theta = \pi - \Delta\theta,\ \theta = \pi\ (\Lambda_0 < x < \Lambda)$. We eliminate values of the functions at nodes outside the contour by using the boundary conditions (1.2) and (1.3) under symmetry conditions /12/

$$[w^{\boldsymbol{\cdot}} = F^{\boldsymbol{\cdot}} = w^{\boldsymbol{\cdot\cdot\cdot}} = F^{\boldsymbol{\cdot\cdot\cdot}}]_{\theta=0,\,\pi} = 0$$

We hence obtain a system of $K = 2(N+1)(M+1)$ non-linear difference equations from (1.1)-(1.3), which we write in the operator mode

$$P(Y,\ p) = 0,\ P = (P_1,\ P_2,\ \ldots,\ P_K),\ P\colon E_K \to E_K \tag{2.1}$$

where E_K is a Euclidean space of dimension K.

For $p = p_0$ let the solution $Y(p_0)$ of system (2.1) be known. We will calculate $Y(p_0 + \Delta p)$ by using the Newtonian iterations

$$y_\gamma(p_0 + \Delta p) = y_\gamma(p_0) + \sum_{l=1}^{t} \delta y_\gamma^{(l)} \tag{2.2}$$
$$\delta y_\gamma{}^{(l)} = (\delta w_{0\gamma}{}^{(l)},\ \delta F_{0\gamma}{}^{(l)},\ \delta w_{1\gamma}{}^{(l)},\ \delta F_{1\gamma}{}^{(l)},\ \ldots,\ \delta w_{N\gamma}^{(l)},\ \delta F_{N\gamma}^{(l)}),$$
$$\gamma = 0,\ 1,\ \ldots,\ M$$

where t is the given number of the iteration and $\delta y_\gamma{}^{(l)}$ is the increment of the vector y_γ at the l-th iteration. These increments (along the rays) are found for $m = 1,\ 2,\ \ldots,\ t$ from the system of linear algebraic equations

$$(P_Y')\,[Y^{(m)},\ p_0 + \Delta p]\,\delta Y^{(m)} = -P\,[Y^{(m)},\ p_0 + \Delta p] \tag{2.3}$$
$$Y^{(m)} = (y_0^{(m)},\ y_1^{(m)},\ \ldots,\ y_M^{(m)}),\ \delta Y^{(m)} = (\delta y_0^{(m)},\ \delta y_1^{(m)},\ \ldots,\ \delta y_M^{(m)})$$
$$y_\gamma^{(1)} = y_\gamma(p_0),\quad y_\gamma^{(r)} = y_\gamma(p_0) + \sum_{l=1}^{r-1} \delta y_\gamma^{(l)},\quad r = 2, 3, \ldots$$

Here $P_Y'(a, p)$ is the Fréchet derivative on the element $a \in E_K$. The linear system (2.3) has the form

$$\begin{gathered} C_0\delta y_0 + B_0\delta y_1 + A_0\delta y_2 = d_0 \\ D_1\delta y_0 + C_1\delta y_1 + B_1\delta y_2 + A_1\delta y_3 = d_1 \\ E_\gamma\delta y_{\gamma-2} + D_\gamma\delta y_{\gamma-1} + C_\gamma\delta y_\gamma + B_\gamma\delta y_{\gamma+1} + A_\gamma\delta y_{\gamma+2} = d_\gamma,\ \gamma = 2, 3, \ldots, \\ M-2 \\ E_{M-1}\delta y_{M-3} + D_{M-1}\delta y_{M-2} + C_{M-1}\delta y_{M-1} + B_{M-1}\delta y_M = d_{M-1} \\ E_M\delta y_{M-2} + D_M\delta y_{M-1} + C_M\delta y_M = d_M \end{gathered} \tag{2.4}$$

The superscript l is omitted on the increments $\delta y_\gamma^{(l)}$ in (2.4). The matrix of system (2.4) has a five-diagonal block structure. The matrices $A_\gamma, B_\gamma, C_\gamma, D_\gamma, E_\gamma$ have dimensions $2(N+1) \times 2(N+1)$, where the matrices A_γ, E_γ are diagonal B_γ, D_γ are seven-diagonal, and C_γ are nine-diagonal. The vectors $d_0, d_1, \ldots, d_M$ defined by the right side have the dimensionality $2(N+1)$. Because of their awkwardness the expressions for the matrix elements of the system (2.4) are not presented.

We seek the solution of system (2.4) by matrix factorization formulas in the form /18/

$$\begin{gathered} \delta y_\gamma = U_\gamma\delta y_{\gamma+1} + V_\gamma\delta y_{\gamma+2} + s_\gamma,\ \gamma = 0, 1, \ldots, M-2 \\ \delta y_{M-1} = U_{M-1}\delta y_M + s_{M-1} \end{gathered} \tag{2.5}$$

To determine the factorization matrices U_γ, V_γ we obtain the formulas

$$\begin{gathered} U_0 = -C_0^{-1}B_0,\ V_0 = -C_0^{-1}A_0,\ s_0 = C_0^{-1}d_0 \\ R_1 = D_1U_0 + C_1,\ U_1 = -R_1^{-1}(D_1V_0 + B_1),\ V_1 = -R_1^{-1}A_1 \\ s_1 = R_1^{-1}(d_1 - D_1s_0),\ R_\gamma = [E_\gamma(U_{\gamma-2}U_{\gamma-1} + V_{\gamma-2}) + D_\gamma U_{\gamma-1} + C_\gamma] \\ U_\gamma = -R_\gamma^{-1}[(E_\gamma U_{\gamma-2} + D_\gamma)V_{\gamma-1} + B_\gamma],\ V_\gamma = -R_\gamma^{-1}A_\gamma \\ s_\gamma = R_\gamma^{-1}[d_\gamma - E_\gamma(U_{\gamma-2}s_{\gamma-1} + s_{\gamma-2}) - D_\gamma s_{\gamma-1}] \end{gathered} \tag{2.6}$$

We first determine $U_\gamma, V_\gamma, s_\gamma$ from (2.6) for $2 \leqslant \gamma \leqslant M-2$, we then find U_{M-1}, s_{M-1} and $\delta y_M = s_M$. We later calculate δy_γ successively for $\gamma = M-1, M-2, \ldots, 0$ by reversing the factorization path by means of (2.5).

The iterations are performed until the inequality

$$\begin{gathered} \max_\gamma |\delta y_\gamma^{(m)}|_0 \leqslant \varepsilon_0 \max_\gamma \Big(\Big|\sum_{l=1}^{m} \delta y_\gamma^{(l)}\Big|_0\Big),\quad m \leqslant t \\ |\delta y_\gamma^{(l)}|_0 = \max\,(|\delta w_{0\gamma}^{(l)}|,\ |\delta F_{0\gamma}^{(l)}|,\ \ldots,\ |\delta w_{N\gamma}^{(l)}|,\ |\delta F_{N\gamma}^{(l)}|), \\ 0 \leqslant \gamma \leqslant M \end{gathered} \tag{2.7}$$

is satisfied.

In an analogous manner, the solution is constructed for the next steps in the motion along the parameter p for given values of ε_0 and t. If the iteration process (2.2)-(2.7) does not converge after t iterations, then the step Δp is halved and the process is repeated from the point p_0. The value of the CL p^* is determined by using an energy criterion for the buckling of conservative elastic systems. The analogue of this criterion for finite-difference equations /19/ results in evaluation of p^* as the least positive root of the equation

$$\det(C_0)\prod_{\gamma=1}^{M} \operatorname{sign}(\det(R_\gamma)) = 0 \tag{2.8}$$

where C_0 and R_γ are matrices from (2.4) and (2.6).

In conformity with the above algorithm, a numerical program was realized on a computer for which the correctness of its operation was confirmed by comparing the upper CL obtained with the results obtained by others.

Values of the upper CL $p^*(\varepsilon) = 0.754$, 0.712; 0.679, respectively, were calculated for $\varepsilon = 0.01$, 0.03, 0.05, for $\Lambda = 6$, $\nu = 0.33$ for an isotropic spherical shell, closed at the apex, and subjected to a pressure distributed as $T = 4p(1 + \varepsilon\sin\theta)$. These results are obtained on a finite-difference 15x10 mesh that corresponds to the partition into $N = 15$ equal intervals along the radial coordinate and $M = 10$ equal intervals along the angular coordinate θ, where $\pi/2 \leqslant \theta \leqslant 3\pi/2$. The values obtained for $p^*(\varepsilon)$ agree well with results known earlier: the quantity $p^*(0.01)$ is 3% less than $p^*(0)$ /20/ while the quantity $p^*(0.05)$ differs by 6-9% from the value of the upper CL found /11/ by using another scheme of the finite-difference method.

For the same spherical shell subjected to an external pressure distributed uniformly over just half its surface ($T(p, x, \theta) \equiv T_1(p, x, \theta) = 4p$ for $0 \leqslant \theta \leqslant \pi$ and $T_1(p, x, \theta) = 0$ for $\pi < \theta < 2\pi$) or just over a quarter of the surface ($T(p, x, \theta) \equiv T_2(p, x, \theta) = 4p$ for $0 \leqslant \theta \leqslant \pi/2$ and $T_2(p, x, \theta) = 0$ for $\pi/2 < \theta < 2\pi$), the results for the upper CL p^* are given in Table 1 (column A) for $\Lambda = 6$, $\nu = 0.33$ together with the results obained by others; the number of partitions N along the radial coordinate and M along the angular coordinate are also presented for half (a quarter) of a spherical shell. It is seen that the results of this paper and those of previous authors diverge by 5-7% in the case of the load T_2 while the divergence increases in the case of the load T_1.

Table 1

Load	A	$N \times M$	[11]	[13]	[9]	[15]
T_1	0.566	22×8	0.68–0.72	0.665	0,66	0.681
T_2	0.531	30×17	–	0.56	–	0.569

Moreover, the values p_H of the upper CL of the non-axisymmetric buckling of ideal orthotropic spherical shells under uniform external pressure were corroborated. According to the procedure developed in /20, 21/, these values of p_H are determined from linear boundary-value problems /6/

$$
\begin{gathered}
L_n^{(1)}(w_n, f_n) \equiv A_n(S_1, S_2, S_3, w_n) - \frac{1}{x}\psi w_n'' - \psi' l_{2,n} w_n + \\
\frac{1}{x}(\theta_* + \beta) f_n'' + (\theta_* + \beta)' l_{2,n} f_n = 0, \quad l_{2,n} w_n = \frac{1}{x} w_n' - \frac{n^2}{x^2} w_n \\
L_n^{(2)}(w_n, f_n) \equiv A_n(S_4, S_5, S_6, f_n) - \frac{1}{x}(\theta_* + \beta) w_n'' - (\theta_* + \beta)' l_{2,n} w_n = 0 \\
A_n(S_1, S_2, S_3, w_n) = S_1\left(w_n^{\mathrm{IV}} + \frac{2}{x} w_n^{\mathrm{III}}\right) - S_2\left[\frac{1}{x^2} w_n'' - \frac{1}{x^3} w_n' - \right. \\
\left. \frac{n^2(n^2 - 2)}{x^4} w_n\right] - S_3 \frac{n^2}{x^2}\left(w_n'' - \frac{1}{x} w_n' + \frac{1}{x^2} w_n\right), \quad (\;)' = \frac{d}{dx}(\;), \; \theta_* = z_*' \\
[f_n = f_n' = \Gamma_{3,n} w_n = \Gamma_{4,n} w_n]_{x=\Lambda_0} = 0 \\
[w_n = w_n' = \Gamma_{1,n} f_n = \Gamma_{2,n} f_n]_{x=\Lambda} = 0 \\
\Gamma_{1,n} f_n = f_n'' + S_7 l_{2,n} f_n, \quad \Gamma_{2,n} f_n = x f_n''' - S_7\left[(n^2 + 1)\frac{1}{x} f_n' - 2\frac{n^2}{x^2} f_n\right] - \\
S_8 \frac{n^2}{x^2}(x f_n' - f_n) - S_9 l_{2,n} f_n \\
\Gamma_{3,n} w_n = w_n'' + S_{10} l_{2,n} w_n, \quad \Gamma_{4,n} w_n = x w_n''' + w_n'' - S_{11} l_{2,n} w_n - \\
S_{12} \frac{n^2}{x^2}(x w_n' - w_n), \quad n = 1, 2, \ldots
\end{gathered}
\tag{2.9}
$$

The functions $\beta(p, x), \psi(p, x)$ are determined from the non-linear boundary-value problem

$$
\begin{gathered}
S_1(x\beta'' + \beta') - S_2 \frac{1}{x}\beta - (\theta_* + \beta)\psi + \varphi(p, x) = 0 \\
S_4(x\psi'' + \psi') - S_5 \frac{1}{x}\psi + \theta_*\beta + \frac{1}{2}\beta^2 = 0 \\
[x\beta' + S_{10}\beta = \psi]_{x=\Lambda_0} = 0, \quad [\beta = x\psi' + S_7\psi]_{x=\Lambda} = 0 \\
\beta = -w^{*\prime}, \quad \psi = F^{*\prime}, \quad \varphi(p, x) = \int_{\Lambda_0}^{x} T(p, \tau)\tau\, d\tau
\end{gathered}
\tag{2.10}
$$

Here the p_H are determined from the formula $p_H = \min_n p_n$, where p_n are the least eigenvalues and (w_n, f_n) are their corresponding vector eigenfunctions of the boundary-value problem (2.9) and (2.10). Note that system (2.9) and (2.10) is derived from (1.1)-(1.3) as a result of linearization with respect to the axisymmetric equilibrium (w^*, F^*) and subsequent expansion of the solution in cosines of multiple arcs.

The results of computations by both methods are practically in agreement for $T(p, x) = 4p$. For instance, for $\Lambda = 6$, $k_1 = 1.5$, $k_2 = 3$, $\nu_1 = 0.25$, $r_0 \equiv a_0/a = 0.1$, $\nu = 0.33$ $p^* = 0.497$ according to (2.2)-(2.8), while $p_H \equiv p_2 = 0.499$ according to (2.9) and (2.10). If $r_0 = 0.25$ and the remaining shell parameters are the very same, then $p^* = 0.337$, $p_H \equiv p_2 = 0.344$. Both results are determined for p^* by a finite-difference mesh of $N \times M = 22 \times 8$ and $\varepsilon_0 = 10^{-4}$ in (2.7).

Remark. The system of non-linear difference Eqs.(2.1) can be considered with respect to a $2(M+1)(N+1)$-dimensional column vector

$$Y = (y_0^*, y_1^*, \ldots, y_N^*),\ y_\alpha^* = (w_{\alpha 0}, F_{\alpha 0}, \ldots, w_{\alpha M}, F_{\alpha M}),\ \alpha = 0, 1, \ldots, N \tag{2.11}$$

that is formed by a set of values of the function pairs w, F on the arcs $x = x_\alpha$ at the mesh nodes. The algorithm elucidated above was also realized for solving the non-linear system (2.1) for the vector Y from (2.11). Formulas are obtained here that are analogous to (2.2)-(2.8) but with $y_\gamma, \delta y_\gamma, M$ and N replaced, respectively, by $y_\alpha^*, \delta y_\alpha^*, N, M$. For $N > M$ such a method of solution is more economical as compared with that elucidated in Sect.2 since matrices of the dimensionality $2(M+1) \times 2(M+1)$ are used in (2.6) in place of matrices of the dimension $2(N+1) \times 2(N+1)$. Note that in the case of an isotropic spherical shell closed at the apex, a variational-difference method in combination with the procedure of continuation in the load parameter and Newton's method was used earlier in /12/ to determine the state of stress and strain in the precritical state and to calculate the values of the upper CL. The system of non-linear difference equations obtained in /12/ was solved for the vector Y in (2.11). It turns out that the linearized system of equations in the vector Y in /12/ and system (2.4) have an identical structure.

3. ***Modification of the finite-difference method for shell analysis in the post-critical stage***. The algorithm (2.2)-(2.7) described in Sect.2 cannot possibly be used to continue the solution in the post-critical domain since condition (2.8) is satisfied at the critical point p^*. To construct the solution in the post-critical stage we use the ideas of the adjustment method /21, 22/. Assuming the point p^* to be the limit, we replace motion in the parameter p in its neighbourhood by motion in the parameter $q = w_{jk}$, where j, k are indices of the mesh node satisfying the conditions $1 \leqslant j \leqslant N-1,\ 1 \leqslant k \leqslant M-1$. In this case the vector

$$Z = (z_0, z_1, \ldots, z_M),\ z_\gamma = (w_{0\gamma}, F_{0\gamma}, \ldots, w_{N\gamma}, F_{N\gamma})$$

$$z_k = (w_{0k}, F_{0k}, \ldots, w_{j-1,k}, F_{j-1,k}, p, F_{jk}, w_{j+1,k}, F_{j+1,k}, \ldots, w_{N,k}, F_{N,k})$$

is to be determined instead of Y in system (2.1), where γ takes all integer values between 0 and m, except k. We calculate the values of z_γ by using the Newtonian iterations

$$z_\gamma(q_0 + \Delta q) = z_\gamma(q_0) + \sum_{l=1}^{t} \delta z_\gamma^{(l)}$$

where δz_γ is written as δy_γ in (2.2) but with δw_{jk} replaced by δp. Here $q_0 = w_{jk}$ is a known value while Δq is the step in the motion in the new parameter. The increment $\delta z_\gamma^{(l)}$ are determined from the system of linear equations

$$(P_Z')\,[Z^{(m)}, q_0 + \Delta q]\,\delta Z^{(m)} = -P\,[Z^{(m)}, q_0 + \Delta q],\ z_\gamma^{(1)} = z_\gamma(q_0)$$

$$z_\gamma^{(r)} = z_\gamma(q_0) + \sum_{l=1}^{r-1} \delta z_\gamma^{(l)},\quad \delta Z^{(m)} = (\delta z_0^{(m)}, \delta z_1^{(m)}, \ldots, \delta z_M^{(m)}),\quad r \geqslant 2$$

which has the following form

$$(L + \Omega)\,\delta Z^{(m)} = d,\ d = (d_0, d_1, \ldots, d_M) \tag{3.1}$$

where L is a five-diagonal block matrix with a structure analogous to the matrix of system (2.4). For a fixed value of k satisfying the condition $2 \leqslant k \leqslant M-2$ all the matrices and vectors, with the exception of $A_{k-2}, B_{k-1}, C_k, D_{k+1}, E_{k+2}, d_{k+l}\ (l = -2, -1, 0, 1, 2)$ are identical with the corresponding matrices and vectors of the system (2.4). The block matrix Ω consists of $M+1$ block columns whose elements are matrices of dimensions $2(N+1) \times 2(N+1)$. The non-zero matrices exist here just in the $(k+1)$-th block column $\omega = (\Omega_0, \Omega_1, \ldots, \Omega_M)$. The presence of this column does not allow direct application of the matrix factorization method (2.5) and (2.6) to system (3.1).

The solution of system (3.1) using the matrix factorization method is constructed successfully if auxiliary unknown vectors u_γ and matrices $\Phi_\gamma\ (\gamma = 0, 1, \ldots, M)$ are introduced by means of the substitutions

$$\delta z_\gamma = u_\gamma - \Phi_\gamma \delta z_k,\ \gamma = 0, 1, \ldots, M \tag{3.2}$$

Here u_γ and Φ_γ are determined, respectively, from the system of scalar and matrix equations

$$\begin{gathered} Lu = d,\ u = (u_0, u_1, \ldots, u_M) \\ L\Phi = \omega,\ \Phi = (\Phi_0, \Phi_1, \ldots, \Phi_M) \end{gathered} \tag{3.3}$$

The first system in (3.3) is solved by the matrix factorization method by means of (2.5) and (2.6) with δy_γ replaced by u_γ. The second system in (3.3) is solved by using the matrix analogue of (2.5) and (2.6) with the vectors $\delta y_\gamma, s_\gamma, d_\gamma$ replaced by the matrices $\Phi_\gamma, G_\gamma, \Omega_\gamma$.

The vectors u_γ and the matrices Φ_γ are determined for all subscripts γ by solving system (3.3). Furthermore, by setting $\gamma = k$ we obtain $\delta z_k = (E + \Phi_k)^{-1} u_k$ from (3.2), where E is the unit matrix. Now, applying (3.2), we find δz_γ for the remaining subscripts γ.

Note that the method considered, of inserting auxiliary unknown vectors and matrices, can be extended to solving systems of the form (3.1) when the matrix Ω contains additional non-zero block columns besides the $(k+1)$-th block column. For example, we shall seek the solution in the presence of the $(l+1)$-th $(l \neq k)$ non-zero column in the form

$$\delta z_\gamma = u_\gamma - \Phi_\gamma \delta z_k - \Pi_\gamma \delta z_l$$

Computation of the state of stress and strain of a spherical shell under non-axisymmetric deformations in the post-critical stage requires a considerable amount of electronic computer time and memory. These increase sharply as the values of the parameter Λ increase.

In the case of axisymmetric deformation, the realization of the algorithm is simplified since we have the boundary-value problem (2.10) in place in the system (1.1)-(1.3), while we obtain a tridiagonal block matrix with matrix elements of dimensions 2x2 in place of the five-diagonal block matrix with matrix elements of dimensions $2(N+1) \times 2(N+1)$ in the linearized systems of equations of the form (3.1).

We will consider the uniform mesh $x_i = \Lambda_0 + ih$ $(i = 0, 1, \ldots, N)$ on the segment $[\Lambda_0, \Lambda]$ with two nodes outside the contour $x_{-1} = \Lambda_0 - h$, $x_{N+1} = \Lambda + h$, where $h = (\Lambda - \Lambda_0)/N$, N is the number of partitions. We introduce the mesh vector-function

$$Y = (y_{-1}, y_0, \ldots, y_{N+1}),\ y_i = (\beta_i, \psi_i),\ i = -1, 0, \ldots, N+1 \tag{3.4}$$

in it formed by a set of values of the pair of functions β, ψ at the mesh nodes x_i. Replacing the derivatives of the functions with respect to x by central finite-difference formulas, we obtain a system of non-linear difference equations of the form (2.1) from (2.10), where E_k is a Euclidean space of dimensions $K = 2(N+3)$.

In this case system (2.1) is solved by using (2.2) and (2.3) in which

$$\delta Y = (\delta y_{-1}, \delta y_0, \ldots, \delta y_{N+1}),\ \delta y_i = (\delta\beta_i, \delta\psi_i),\ i = -1, 0, \ldots, N+1$$

and Y is defined in (3.4). The corresponding linear system, analogous to system (2.3), has the tridiagonal block structure /2/

$$G_0 \delta y_1 + H_0 \delta y_0 - G_0 \delta y_{-1} = 0 \tag{3.5}$$

$$C_i \delta y_{i-1} + B_i \delta y_i + A_i \delta y_{i+1} = d_i,\ i = 0, 1, \ldots, N \tag{3.6}$$

$$-G_N \delta y_{N-1} + H_N \delta y_N + G_N \delta y_{N+1} = 0 \tag{3.7}$$

The matrices A_i, B_i, C_i $(i = 0, 1, \ldots, N)$, G_j, H_j $(j = 0, N)$ have the dimensions 2x2. System (3.5)-(3.7) is solved by the matrix factorization method /2/.

On approaching the limit point p^* the motion in the parameter p is replaced by motion in the parameter $q = \beta_k \equiv \beta(x_k)$, where k is any integer between 0 and N. Note that for a numerical realization of the algorithm, the subscript k was assumed to be equal to the number of the node at which the function β has the greatest change.

After substitution of the new parameter, the linear system

$$-G_0 \delta z_{-1} + H_0 \delta z_0 + G_0 \delta z_1 + \Omega_{-1} \delta z_k = 0 \tag{3.8}$$

$$C_i \delta z_{i-1} + B_i \delta z_i + A_i \delta z_{i+1} + \Omega_i \delta z_k = d_i,\ i = 0, 1, \ldots, N \tag{3.9}$$

$$-G_N \delta z_{N-1} + H_N \delta z_N + G_N \delta z_{N+1} + \Omega_{N+1} \delta z_k = 0 \tag{3.10}$$

$$Z = (z_{-1}, z_0, \ldots, z_{N+1}),\ \delta Z = (\delta z_{-1}, \delta z_0, \ldots, \delta z_{N+1})$$

$$z_i = (\beta_i, \psi_i),\ \delta z_i = (\delta\beta_i, \delta\psi_i),\ i = -1, 0, 1, \ldots, N+1,\ i \neq k$$

$$z_k = (p, \psi_k),\ \delta z_k = (\delta p, \delta\psi_k)$$

$$\Omega_{-1} = \Omega_{N+1} = 0, \quad \Omega_i = \begin{Vmatrix} a_i & 0 \\ 0 & 0 \end{Vmatrix}, \quad a_i = \frac{\partial\varphi(x_i, p)}{\partial p}, \quad i = 0, 1, \ldots, N$$

is obtained that can be solved by formulas analogous to (3.2) and (3.3).

We will present a more-efficient method of solving system (3.8)-(3.10) by using the presence of one non-zero element in the matrices Ω_i. Let $3 \leqslant k \leqslant N-3$. First we eliminate δz_{-1} from (3.9) for $i = 0$ by using (3.8). Furthermore, we subtract $2i+3$ rows multiplied by a_i/a_{i+1} from the $2i+1$ scalar rows for $i = 0, 1, \ldots, k-3$ in (3.9). Then we subtract $2i+1$ rows multiplied by a_i/a_{i-1} from $2i+3$ scalar rows for $i = N, N-1, \ldots, k+3$. We consequently have the linear system

$$\begin{gathered}
B_0\delta z_0 + A_0\delta z_1 + L_0\delta z_2 = d_0 \\
C_i\delta z_{i-1} + B_i\delta z_i + A_i\delta z_{i+1} + L_i\delta z_{i+2} = d_i,\ i = 1,\ 2,\ \ldots,\ k-2 \\
C_i\delta z_{i-1} + B_i\delta z_i + A_i\delta z_{i+1} = d_i,\ i = k-1,\ k,\ k+1 \\
M_i\delta z_{i-2} + C_i\delta z_{i-1} + B_i\delta z_i + A_i\delta z_{i+1} = d_i,\ i = k+2, \\
k+3,\ \ldots,\ N \\
-G_N\delta z_{N-1} + H_N\delta z_N + G_N\delta z_{N+1} = 0
\end{gathered} \tag{3.11}$$

Here A_i, B_i, C_i, L_i and M_i are new matrices obtained as a result of the above algebra. System (3.11) has a five-diagonal block matrix with the structure shown in the figure (the crosses denote the non-zero matrices of the system while the values of i in the column at the left indicate the subscript ascribed to the block rows of system (3.11)). Such a structure enables a solution to be sought by matrix factorization formulas in the form

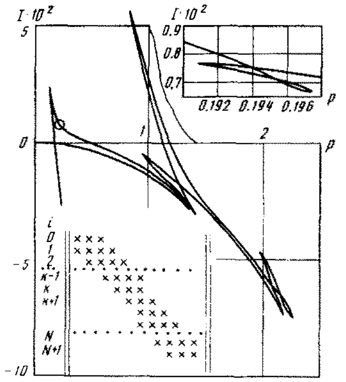
$$\begin{gathered}
\delta z_i = U_i\delta z_{i+1} + V_i\delta z_{i+2} + s_i,\ i = 0,\ 1,\ \ldots,\ k-2 \\
\delta z_i = Q_i\delta z_{i+1} + s_i,\ i = k-1,\ k,\ \ldots,\ N
\end{gathered}$$

In the cases $k = 0$ and $k = 1$ for $i = N,\ N-1,\ \ldots,\ k+3,$ $2i+1$ rows multiplied by a_i/a_{i-1} are subtracted from the $2i+3$ scalar rows and the linear system obtained is solved by formulas of three-point matrix factorization. The remaining cases are considered analogously. For $k = N, N-1, N-2$ the linear system is solved by the formulas of five-point matrix factorization.

A graph of the dependence on p of the functional

$$\begin{gathered}
I = \frac{1}{2\Lambda^4}\left\{\int\limits_{\Lambda_0}^{\Lambda}\left[S_1x\beta'^2 + S_2\frac{1}{x}\beta^2 + S_4x\psi'^2 + S_5\frac{1}{x}\psi^2 - 4p\,(x^2 - \Lambda_0{}^2)\,\beta\right]dx + \right. \\
\left. S_4S_7\psi^2(\Lambda) - S_1S_{10}\beta^2(\Lambda_0)\right\}
\end{gathered}$$

that is proportional to the potential energy of a uniformly loaded $(T(p, x) = 4p)$ rigidly clamped isotropic spherical shell along the external edge, is presented in the figure for $\Lambda = 10,\ \Lambda_0 = \Lambda/3,$ $\nu = 0.33.$ The section of the curve distinguished by the circle is represented on a magnified scale in the upper right hand corner of the figure.



We note that an algorithm based on reducing the equilibrium equations to a boundary-value problem for a system of first-order equations and by iteration continuation in the numerical parameter being varied by using Newton's method and the method of finite differences was developed /23/ to investigate the axisymmetric post-critical behaviour of geometrically non-linear shells of revolution.

4. ***The upper CL of orthotropic spherical shells with initial imperfections.*** Because of the limitations of the BESM-6 computer memory and speed of response, an analysis of the non-axisymmetric deformation of shells in the precritical stage was successfully performed by using the algorithm constructed in Sect.2 and the upper buckling CL was determined only for small values of the parameter Λ. For an orthotropic truncated spherical shell with non-symmetric initial deflection $\xi\zeta(x, \theta) = \xi\zeta_n(x)\cos n\theta$ subjected to a uniformly distributed external pressure $(T(p, x, \theta) = 4p)$, the computations can be performed even for large values of Λ taking into account the fact that problem (1.1)-(1.3) possesses the property of cyclic symmetry with n axes $\theta_l = \pi l/n\ (l = 0, 1, \ldots, n-1)$, and therefore, it is possible to confine oneself to the construction of a difference mesh in the domain $D_n = \{\Lambda_0 \leqslant x \leqslant \Lambda,\ 0 \leqslant \theta \leqslant \pi/n\}$.

Results of computations of the upper buckling CL p^* are presented in Table 2 for orthotropic spherical shells with initial deflection $\xi\zeta = \xi\,(x - \Lambda_0)\,(x - \Lambda)\cdot\cos n\theta$ for different values of the parameters $\Lambda,\ r_0,\ \xi,\ n$ for $k_1 = 1.5,\ k_2 = 3,\ \nu = 0.33,$ and $\nu_1 = 0.25$. These results are obtained for a number of partitions $N = 22$ along the radial coordinate and $M = 8$ along the angular coordinate in the domain D_n. It is seen that an increase in the amplitude of the initial imperfections as well as an increase in the relative radius of the hole r_0 will result in a reduction in the upper CL.

Table 2

Λ	$r_0\times 10^2$	$\xi\times 10^2$	n	$p^*\times 10^3$	$p_s\times 10^3$	$p_n\times 10^3$
6	2	2	2	473	457	548
6	2	4	2	428	403	548
6	10	1	2	458	456	499
6	10	2	2	433	430	499
6	25	2	2	319	320	344
6	25	4	2	307	307	344
7	1	1	3	463	450	524
7	1	2	3	423	407	524
7	1	4	3	369	338	524

Nevertheless, the algorithm described enables one to estimate the effectiveness of the use of the theory of buckling and initial post-critical behaviour /1-4, 6, 24/ to determine the upper CL of spherical shells. According to this theory, when there are small harmonic imperfections in the shell shape in the azimuthal direction and analogous load deviations from a uniformly distributed one, the bifurcation point p_0 transfers under unimodal buckling to the limit point p_s that is determined by the formula /1-4/

$$(p_s - p_0)^{3/2} = {}^3/_2\,|\,\xi d\,|\,\sqrt{-3b},\quad |\,\xi\,| \ll 1 \tag{4.1}$$

which is a result of solving the system of equations

$$X_1 \equiv L_{300}\mu_1^3 + L_{110}(p - p_0)\,\mu_1 + L_{001}\xi + \ldots = 0,\quad \partial X_1/\partial\mu_1 = 0 \tag{4.2}$$

The first of Eqs.(4.2) is a bifurcation equation that is written down to an accuracy of higher-order quantities, while the second equation is the buckling condition. Here p_0 is the eigenvalue of the boundary-value problem (2.9) and (2.10) that has the eigenvector-function $(w_n, f_n)\cos n\theta$. The parameters b and d in (4.1) are determined from the formulas

$$b = -L_{300}L_{110}^{-1},\quad d = -L_{001}L_{110}^{-1},\quad L_{110} \neq 0 \tag{4.3}$$

$$L_{300} = -\frac{4\pi}{\Lambda^4}\int\limits_{\Lambda_*}^{\Lambda}\left[\beta_1 g_1 - \alpha_1 g_2 - \frac{1}{2}x\,(H_1t_1 - H_2t_2)\right]dx$$

$$L_{110} = \frac{4\pi}{\Lambda^4}\int\limits_{\Lambda_*}^{\Lambda}\frac{\partial\varphi(p, x)}{\partial p}\beta_1 dx,\quad L_{001} = \frac{1}{\Lambda^4}\int\limits_0^{2\pi}\int\limits_{\Lambda_0}^{\Lambda}\zeta(x, \theta)\cos n\theta\left[\frac{n^2}{x}(\psi' w_n -\right.$$

$$\left.\beta' f_n) - w_n''\psi - w_n'\psi' + f_n''\beta + f_n'\beta'\right]dx\,d\theta$$

The functions in (4.3) are found from the linear boundary-value problems

$$S_1(x\beta_1')' - S_2\frac{1}{x}\beta_1 - \psi\beta_1 - (\theta_* + \beta)\,\alpha_1 = g_1(x) \tag{4.4}$$

$$S_4(x\alpha_1')' - S_5\frac{1}{x}\alpha_1 + (\theta_* + \beta)\,\beta_1 = g_2(x)$$

$$[x\beta_1' + S_{10}\beta_1 = \alpha_1]_{x=\Lambda_0} = 0,\quad [\beta_1 = x\alpha_1' + S_7\alpha_1]_{x=\Lambda} = 0$$

$$g_1(x) = \frac{1}{2}\left[n^2\left(\frac{1}{x}w_n f_n\right)' - w_n' f_n'\right],\quad g_2(x) = \frac{1}{4}\left[n^2\left(\frac{w_n^2}{x}\right)' - (w_n')^2\right]$$

$$L_{2n}^{(1)}(H_1, H_2) = t_1(x),\quad L_{2n}^{(2)}(H_1, H_2) = t_2(x)$$

$$[H_2 = H_2' = \Gamma_{3,\,2n}\ H_1 = \Gamma_{4,\,2n}H_1]_{x=\Lambda_0} = 0$$

$$[H_1 = H_1' = \Gamma_{1,\,2n}H_2 = \Gamma_{2,\,2n}H_2]_{x=\Lambda} = 0$$

$$2xt_1 = [w_n, f_n, n] + [f_n, w_n, n] + 2\frac{n^2}{x}[w_n][f_n],\quad 2xt_2 = [w_n, w_n, n] -$$

$$\frac{n^2}{x}[w_n][w_n],\quad [w_n, f_n, n] = w_n''\left(f_n' - \frac{n^2}{x}f_n\right),\quad [w_n] = w_n' - \frac{1}{x}w_n$$

The upper CL of problem (1.1)-(1.3) can be obtained by the formula for p_s from (4.1)-(4.3) over a wide range of variation of the parameter Λ since for this only the boundary-value problem (2.9), (2.10) and (4.4) must be solved for systems of ordinary differential equations.

The effectiveness of using (4.1) to evaluate the upper CL for certain values of the parameters Λ, r_0, ξ, n is illustrated by Table 2, in which values of p^* are presented together with p_s for the upper CL evaluated by means of (2.2)-(2.8). These values differ by not more than 9%. To estimate the influence of the initial imperfections on the reduction of the CL, values of the critical loads p_n of non-axisymmetric buckling of an ideal shell in a form proportional to the harmonic $\cos n\theta$ are represented in the last column of Table 2.

5. The upper CL of isotropic spherical shells under uniform external pressure. The results of calculating the upper CL by non-axisymmetric theory /20/ are in good agreement with experimental data /7, 8/ and were confirmed /11, 21/. Meanwhile, experimental values of the critical pressures obtained by Parmerter, Ivan-Ivanovskii, et al., Tillmann (see /7, 25/), Pogorelov /26/, Sunakova and Isida /27/, and Babenko and Prichko /28/ turned out to be somewhat higher than the theoretical results /20/. The discrepancies obtained were recently explained in /29/. Conclusions were drawn in /30/, on the basis of the results in /10, 29/, concerning the complete agreement between the theory of large deflections /20/ and the experiment for a rigidly clamped spherical shell subjected to uniform external pressure. Moreover, it was established /30/ that the discrepancy between the theoretical values of the upper CL and the corresponding experimental data as well as the spread in the experimental data themselves can be satisfactorily explained if the imperfections are taken into account accurately.

The equilibrium equations of isotropic spherical shells closed at the apex with initial deflection subjected to an external transverse load can be writen in dimensionless variables in the form of a system of equations with boundary conditions

$$\begin{gathered}\Delta^2 w-[w-z,\ F]=T(p,\ x,\ \theta),\quad \Delta^2 F-\left[z-\frac{1}{2}w,\ w\right]=0 \\ \Delta w=l_1 w+l_2 w,\quad z(x,\theta)={}^1/_2(\Lambda^2-x^2)+\xi\zeta(x,\ \theta) \\ 0\leqslant x\leqslant\Lambda,\quad 0\leqslant\theta<2\pi \\ \Gamma_1 F=F''-\nu l_2 F,\quad \Gamma_2 F=xF'''+\nu\left(\frac{1}{x}F'-\frac{1}{x}F'^{\cdot\cdot}+\frac{2}{x^2}|F^{\cdot\cdot}\right)+ \\ 2(1+\nu)\left(\frac{1}{x}F'^{\cdot\cdot}-\frac{1}{x^2}|F^{\cdot\cdot}\right)-l_2F \\ [w=w'=\Gamma_1F=\Gamma_2F]_{x=\Lambda}=0\end{gathered}\tag{5.1}$$

System (1.1)-(1.4) changes into the boundary-value problem (5.1) for $E_1=E_2=E$, $\nu_1=\nu_2=\nu$, $G={}^1/_2E/(1+\nu)$. For this case, we should set

$$S_1=S_2=S_4=S_9=1,\quad S_3=S_6=2,\quad S_7=-\nu,\quad S_8=2(1+\nu)\tag{5.2}$$

in (2.9), (2.10), (4.1)-(4.4) to determine p_s by Koiter's theory.

Changes associated with the conditions /12/ at the shell pole were substituted into the system of finite-difference Eqs.(2.1) for numerical computations of p^* in the case of boundary-value problem (5.1) and (5.2). Note that problem (5.1) was reduced /11, 16/ to a system of second-order equations solved by successive approximations by using the change of variables $\Delta w=\varphi_1$, $\Delta F=\varphi_2$. A finite-difference method with a nine-point pattern was used here to solve linear boundary-value problems at each step. The change of variables mentioned in the algorithm of Sect.2 is inefficient since the volume of calculations increases considerably.

Results of computations for p^* and p_s are represented in Table 3 for the upper CL of uniformly loaded spherical shells with the initial deflection

$$\xi\zeta=\xi x^m(x-\Lambda)\cos n\theta\tag{5.3}$$

for $\Lambda=6$, $\Lambda=7$ and $m=1$. Values of the bifurcation points p_n corresponding to the CL of the buckling of an ideal spherical shell in the intrinsic form $(w_n, f_n)\cos n\theta$ are presented in the last column. For $m=1$ the values of p^* and the values of p_s differ by not more than 2.2% for $|\xi|\leqslant 0.02$; as $|\xi|$ increases this discrepancy increases and reaches 13% for $\xi=0.1$.

The results presented in Table 3 confirm that the presence of initial imperfections is the reason for the reduction of the upper CL and, moreover, its values can be determined by Koiter's formula (4.1) not at the least bifurcation point but at the next if the initial deflection components proportional to the harmonic of the intrinsic form of this second bifurcation point is predominant. The upper CL of non-axisymmetric buckling or an ideal shell for $\Lambda=6$ equals the value of the least bifurcation point $p_2=0.772$ according to /20/. The next bifurcation point $p_3=0.827$ is located after the point p_2. Assuming $p_0=p_2$, we obtain from (4.1)-(4.3) that the imperfection (5.3) does not influence the CL p_2 for $m=1, n=3$ since $d=0$. Furthermore, assuming $p_0=p_3$ we find $d\neq 0$ and it follows from (4.1) that the bifurcation point p_3 transfers into the limit point $p_s=0.742$ for $\xi=0.01$ and $p_s=0.693$ for $\xi=0.02$. The calculated values of p_s are less than p_2. Therefore, it is not the lowest point of bifurcation p_2 but the next bifurcation point p_3 after it that generates the upper buckling CL of an imperfect shell. Naturally the upper CL is given by (4.1) for $p_0=p_2$ in the case of initial damage (5.3) for $m=1$, $n=2$.

The results of compuations for the initial damage (5.3) are represented in Table 4 for $m=2$, from which it follows that in improvement in the smoothness of the initial imperfection at $x=0$ results in a decrease in the discrepancy between the values of p^* and p_s for identical values of ξ. In particular, it does not exceed 1.6% for $\xi=0.05$ and $\xi=0.1$ for $\Lambda=6$ and does not exceed 3% for $\xi=0.05$ for $\Lambda=7$.

Table 3

Λ	$\xi\times10^2$	n	$N\times M$	$p^*\times10^3$	$p_s\times10^3$	$p_n\times10^3$
5,53	1	1	20×8	702	717	778
5,53	3	1	20×8	637	650	778
5,53	5	1	20×8	586	599	778
6	1	2	10×6	693	689	772
6	2	2	10×6	645	640	772
6	10	2	10×6	453	385	772
6	1	3	12×6	746	742	826
6	2	3	12×6	698	693	826
7	1	3	23×8	654	647	758
7	2	3	23×8	595	582	758
7	1	4	23×8	702	698	810
7	2	4	23×8	644	632	810

Table 4

Λ	$\xi\times10^2$	n	$p^*\times10^3$	$p_s\times10^3$
6	1	2	734	726
6	5	2	647	637
6	10	2	565	557
7	1	3	696	689
7	3	3	624	615
7	5	3	572	557

For $n = 1$ we obtain from the boundary-value problem (2.9) and (5.2) for determining w_1, f_1

$$xY_1'' - Y_1' - \frac{3}{x} Y_1 - \psi Y_1 + (\beta + \theta_*) Y_2 = 0$$

$$xY_2'' - Y_2' - \frac{3}{x} Y_2 - (\beta + \theta_*) Y_1 = 0$$

$$[Y_1 = Y_2]_{x=0} = [Y_1 = xY_2' - \nu Y_2]_{x=\Lambda} = 0$$

$$Y_1 = xw_1' - w_1, \quad Y_2 = xf_1' - f_1$$

The values p_s is found by using (4.1)-(4.4), and (5.2) but with the boundary conditions at $x = \Lambda_0$ replaced by the condition for $x = 0$ /3, 4, 24/, and the functions g_1, g_2, t_1, t_2 replaced by the following:

$$g_1(x) = -\frac{1}{2x^2} Y_1 Y_2, \quad g_2(x) = -\frac{1}{4x^2} Y_1^2$$

$$t_1(x) = \frac{1}{2x^5} (x^2 Y_1 Y_2)', \quad t_2(x) = -\frac{1}{4x^5} (x^2 Y_1^2)'$$

The results of computations of the upper CL p^* and p_s for a spherical shell for $\Lambda = 5{,}53$ and the initial damage (5.3) for $m = n = 1$ subjected to uniform external pressure are presented in Table 3. In this case the values of p^* and p_s differ by not more than 2.3% for $|\xi| \leqslant 0{,}05$.

Consider the problem /11, 21/ of calculating values of the upper CL of an ideal isotropic spherical shell subjected to external loads of the form $T_3(p, x, \theta) = 4p + \eta \cos n\theta$, $T_4(p, x, \theta) = 4p(1 + \varepsilon \cos n\theta)$, $\eta_0 = \varepsilon \equiv 0$ for $x < 10^{-3}$ and $\eta_0 = \eta$, $\varepsilon_0 = \varepsilon$ for $x \geqslant 10^{-3}$, where ε and η are small scalar quantities.

A system of non-linear differential equations with boundary conditions

$$L_0^2 w_0 = L_0 f_0 + \frac{1}{x} (w_0' f_0')' - n^2 R f_n R (w_n - \xi\zeta_n) + \frac{1}{2} f_n'' S (w_n - \xi\zeta_n) + \frac{1}{2} (w_n - \xi\zeta_n)'' S f_n + 4p \tag{5.4}$$

$$L_0^2 f_0 = -L_0 w_0 - \frac{1}{x} w_0' w_0'' + \frac{1}{2} n^2 (R w_n)^2 - \frac{1}{2} w_n'' S w_n + \frac{1}{2} \xi\zeta_n'' S w_n + \frac{1}{2} \xi w_n'' S \zeta_n - \xi n^2 R w_n R \zeta_n$$

$$S w_n = \frac{1}{x} w_n' - \frac{n^2}{x^2} w_n, \quad R w_n = \frac{1}{x} w_n' - \frac{1}{x^2} w_n, \quad \zeta_0 = 0$$

$$w_0 \approx A_0 + B_0 x^2, \quad f_0 \approx C_0 + D_0 x^2 \quad (x \to 0)$$

$$[w_0 = w_0' = \Gamma_{1,0} f_0 = \Gamma_{2,0} f_0]_{x=\Lambda} = 0$$

$$\begin{gathered}
L_n{}^2 w_n = L_n f_n + \frac{1}{x}\left(f_0{}' w_n{}'' + w_0{}' f_n{}''\right) + w_0{}'' S f_n + f_0{}'' S w_n - \\
\xi \frac{1}{x} f_0{}' \zeta_n{}'' - \xi f_0{}'' S \zeta_n + 4p\varepsilon \\
L_n{}^2 f_n = -L_n w_n - \frac{1}{x} w_0{}' w_n{}'' - w_0{}'' S\left(w_n - \xi\zeta_n\right) + \xi \frac{1}{x} w_0{}' \zeta_n{}'' \\
w_n \approx A_n x^n + B_n x^{n+2},\ f_n \approx C_n x^n + D_n x^{n+2} \quad (x \to 0) \\
[w_n = w_n{}' = \Gamma_{1,\,n} f_n = \Gamma_{2,\,n} f_n]_{x=\Lambda} = 0
\end{gathered} \tag{5.5}$$

was derived from (5.1) for solving this problem in /21/, on the basis of the assumption that components with the zero-th and n-th azimuthal harmonics play the main part in the cosine-series expansions of the functions w and F.

Compared with formulas (4.136)-(4.139) in /21/, components were appended here corresponding to the n-th harmonic of the initial deflection in the form of the middle surface. The expressions for $\Gamma_{1,\,n} f_n$, $\Gamma_{2,\,n} f_n$ are written by using (2.9) and (5.2).

The method of adjustment based on reduction to a Cauchy problem and the determination of the adjustment parameters A_0, $B_0, D_0, A_n, B_n, C_n, D_n$ from a system of seven non-linear algebraic equations corresponding to the boundary conditions on the right end, was used to solve the boundary-value problem (5.4) and (5.5) in /21/.

We will describe what, in our opinion, is a simpler method of solving system (5.4) and (5.5). For $p = p_1$ let the vector-function $V_0 = \{w_0(x, p_1), f_0(x, p_1)\}$ be known. Setting $p = p_1 + \Delta p$ and taking V_0 as the initial approximation of the vector-function $\{w_0(x, p), f_0(x, p)\}$, we obtain a linear boundary-value problem in w_n, f_n from (5.5), from which we find the vector-function $U_i = \{w_n{}^{(i)}(x, p), f_n{}^{(i)}(x, p)\}$ for $i = 1$ by finite-difference methods. Now, replacing $\{w_n, f_n\}$ by U_1, we obtain a non-linear boundary-value problem for w_0, f_0 from (5.4) from which we find $V_i = \{w_0{}^{(i)}(x, p), f_0{}^{(i)}(x, p)\}$ for $i = 1$ by using Newtonian iterations and the method of finite differences. Further, using V_1 instead of V_0, we find U_2 from (5.5) and then taking account of U_2 we calculate V_2 from (5.4). We continue the iteration process for $p = p_1 + \Delta p$ until the solution of system (5.4) and (5.5) is found with a given degree of accuracy. After this we obtain $p = p_1 + 2\Delta p$ etc. The method can be carried over to the case of shells of revolution, including conical and cylindrical shells, with small changes in the boundary-value problems (5.4) and (5.5). This same algorithm can be realized analogously by using the adjustment method /4, 21/.

Table 5

Λ	n	$\eta\times10^2$	$p^*\times10^3$	$p_s\times10^3$	$\varepsilon\times10^2$	$p^*\times10^3$	$p_s\times10^3$
5.53	1	3	728	742	1	730	742
5.53	1	9	684	703	3	690	706
5.53	1	15	650	673	5	660	680
6	2	3	749	739	1	729	739
6	2	9	710	704	3	694	706
6	2	15	681	676	5	669	682

Table 6

n	$p^*\times10^3$	$p_s\times10^3$	$p_{BM}\times10^3$	$p_B\times10^3$
2	722	691	716	725
3	699	708	690	695

Table 5 shows the results of numerical computations of the upper CL of ideal isotropic spherical shells subjected to external loads T_3 (the left side of the table) and T_4 (the right side). Values of p^* are calculated by (2.2)-(2.8) and values of p_s by using the Lyapunov-Schmidt method, where the load deviation from hydrostatic is taken into account as in /32/. Values of p_s in the case of the load T_3 were determined by (4.1) but with ξ replaced by η and L_{001} by

$$L_{001}^* = \frac{1}{\Lambda^4} \int_0^{2\pi} \int_0^{\Lambda} \cos^2 n\theta\, w_n x\,dx\,d\theta = \frac{\pi}{\Lambda^4} \int_0^{\Lambda} w_n x\,dx$$

In the case $n = 1$ the expression for L_{001}^* is converted to the form

$$L_{001}^* = -\frac{\pi}{3\Lambda^4} \int_0^{\Lambda} Y_1 x\,dx, \quad Y_1 = x w_1{}' - w_1$$

by using the Dirichlet formula.

We obtain the bifurcation equation for the load T_4 in the form

$$X_2 \equiv L_{300}\mu_1^3 + L_{110}(p - p_0)\mu_1 + 4p\varepsilon L_{001}^* + \ldots = 0$$

where L_{300} and L_{110} are given by (4.3) while the value of p_s is found from the formula

$$(p_s - p_0)^{3/2} = 6p_s \quad |\varepsilon d| \quad \sqrt{-3b} \tag{5.6}$$

which is obtained from the solution of the system

$$X_2 = 0, \ \partial X_2/\partial\mu_1 = 0$$

The simple iteration method was used in solving (5.6). The values of p^* and p_s differ by not more than 3.5% for $|\eta| \leqslant 0.15$ for the load T_3 and by 3.1% for $|\varepsilon| \leqslant 0.05$ for the load T_4.

Results of computations of the upper CL obtained by different methods are shown in Table 6 for an isotropic spherical shell subjected to an external load T_4 for $\Lambda = 7, \nu = 0.33, \varepsilon = 0.02$. In addition to the values of p^* and p_s, results are given for values of p_B /21/ and values of p_{BM} calculated on the basis of the boundary-value problems (5.4) and (5.5) using the modification of the method described above /21/. The results presented illustrate the efficiency of the Lyapunov-Schmidt method and the method of /21/ in the case of buckling in one natural shape.

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ON THE ANALYSIS OF THIN POROUS COATINGS*

E.V. KOVALENKO

A plane contact problem is considered for an elastic layer whose pores are filled with a viscous incompressible fluid. It is shown that in the case of a relatively small layer (coating) thickness its rheological properties can be modelled by equations of the Fuss-Winkler foundation with a bed operator coefficient (the analogue of the hereditary elasticity equations). The case of the impression of a parabolic stamp in a thin porous-elastic coating is investigated in detail. Asymptotic formulas are obtained for the fundamental contact interaction characteristics, namely, the settling of the foundation under the stamp, the contact domain, and the contact pressure, which hold for short and long times.

The experience of producing and using antifriction coatings in modern engineering results in the need to control their structure and functional properties. Among such coatings one should mention primarily porous-elastic coatings whose surface is antifrictional by virtue of its ability to absorb oil and then to release it under loading. Moreover, the theory of the deformation of porous-elastic bodies is convenient for describing a number of features of material production by porous metallurgy methods /1/. The principles of this theory were

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